

Josephson array of mesoscopic objects. Modulation of system properties through the chemical potential

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The phase diagram of a two-dimensional Josephson array of mesoscopic objects (superconducting granules, superfluid helium in a porous medium, traps with Bose-condensed atoms, etc.) is examined. Quantum fluctuations in both the modulus and phase of the superconducting order parameter are taken into account within a lattice boson Hubbard model. Modulating the average occupation number n_0 of the sites in the system (the “number of Cooper pairs” per granule, the number of atoms in a trap, etc.) leads to changes in the state of the array, and the character of these changes depends significantly on the region of the phase diagram being examined. In the region where there are large quantum fluctuations in the phase of the superconducting order parameter, variation of the chemical potential causes oscillations with alternating superconducting (superfluid) and normal states of the array. On the other hand, in the region where the bosons interact weakly, the properties of the system depend monotonically on n_0 . Lowering the temperature and increasing the particle interaction force lead to a reduction in the width of the region of variation in n_0 within which the system properties depend weakly on the average occupation number. The phase diagram of the array is obtained by mapping this quantum system onto a classical two-dimensional XY model with a renormalized Josephson coupling constant and is consistent with our quantum Path-Integral Monte Carlo calculations.

1. INTRODUCTION

Advances in microlithographic techniques have made it possible to create regular arrays of mesoscopic Josephson objects. These systems, which are under active experimental and theoretical study, include superfluid helium in porous media¹⁾ (see Ref. 2 and the literature cited there), lattices of mesoscopic Josephson contacts,^{3,4} and ultrasmall superconducting granules.^{5,6} One interesting physical realization of a Josephson array consists of Josephson junctions in a structure with superfluid ³He created by means of lithography.⁷ Major advances in experiments with Bose condensates of atoms cooled by laser irradiation followed by evaporation^{8–10} suggest that it may be possible to fabricate a Josephson array of close magneto-optical traps with Bose-condensed atoms²⁾ or with clusters of Bose-condensed atoms cooled and localized at the nodes of a system of standing electromagnetic waves. Finally, another remarkable realization of a Josephson array might be a system of Josephson coupled “lakes” of Bose-condensed excitons in single or double quantum wells located at the minima of the random field created by the roughness of the well surfaces, i.e., in “natural” quantum dots,¹² or in an array of artificial quantum dots.

For concreteness, the following discussion considers the example of a system of superconducting mesoscopic granules or Josephson junctions, but the results apply to all the systems mentioned above. We consider a regular array of mesoscopic granules situated on a conducting substrate and separated from it by a thin dielectric layer. A voltage applied to the conducting substrate serves as the chemical potential of the Cooper pairs, which determines the average occupation number n_0 of the granules in the system.^{13,14} For example, in the case of superfluid helium in a porous medium, the chemical potential of the atoms can be varied (because of the contribution of the van der Waals interaction) by changing the thickness of the layer of adsorbed helium.^{15,16} The state of an array of mesoscopic traps with cooled atoms can be controlled by changing the average number of atoms in the system (by, for example, capturing them from an external flow).

The systems being examined here, in which the character of the processes taking place is determined by the boson degrees of freedom, are conveniently described by a lattice boson Hubbard model^{17,18} with the Hamiltonian

$$\hat{H} = \frac{t}{2} \sum_{\langle i,j \rangle} (2a_i^\dagger a_i - a_i^\dagger a_j - a_j^\dagger a_i) + \frac{U}{2} \sum_i (a_i^\dagger a_i)^2 - \mu \sum_i a_i^\dagger a_i \quad (1)$$

In this model site i corresponds to one superconducting granule or pore with helium, to a single trap with a Bose condensate, etc. The operators a_i^\dagger (a_i) are the Bose creation (annihilation) operators of an “effective” boson at site $i = \overline{1, N^2}$ of an $N \times N$ lattice. The first term in the Hamiltonian takes into account the kinetic energy of the particles,

which corresponds to the Josephson tunneling energy and is described by the parameter t . The second term in Eq. (1) describes the interaction of effective bosons at a granule with a characteristic energy $U_i > 0$.

The model (1) is interesting in that it can be used to study the properties of arrays of mesoscopic structures in which the relative fluctuations in the modulus of the superconducting order parameter are large. In this regard, we note that the quantum XY model is justified only when these fluctuations are small,¹⁴ i.e., in the case of arrays of macroscopic granules.

A lattice system (1) with Mott-insulator and superconducting phases at $T = 0$ has been studied both analytically^{17–19} and by computer simulation.^{20,21} In this paper we shall be interested in the properties of the system (1) at finite temperatures, digressing from the interesting phase transitions at $T = 0$.²² In our earlier papers,²³ the investigation was limited to the case of integer (commensurable) populating, in which the average number of bosons per site (granule), $n_0 = \langle a_i^\dagger a_i \rangle$, is a whole number. At $T = 0$, adding even a single particle to an arbitrarily large system changes its properties in a fundamental way. Specifically, a system with a noninteger average number of bosons per granule remains superconducting for arbitrary values of U/t for the interaction between the particles.^{17–22} It is evident that this surprising behavior should be present even in the limit of $T = 0$. In fact, as will be shown below, at finite temperatures the properties of the system vary little within an interval $n_0 = k \pm \delta n_0$ about an integer population, whose width $2\delta n_0$ decreases as the temperature is lowered.

The purpose of this paper is to study the changes in the character of the ordering in an array of granules as the substrate voltage (chemical potential of the effective bosons) is varied. Here we shall not use the simplifying assumption of small relative fluctuations in the modulus of the superconducting order parameter. The results given below correspond to an array of mesoscopic objects for which the root-mean-square fluctuations in the number of particles are comparable to their average number. In Sec. 2 we present results from mean-field calculations. The method used there corresponds to mapping the initial boson model (1) onto an effective classical XY model with a renormalized Josephson coupling constant. To refine the results of the analytical calculations, we use the quantum Path-Integral Monte Carlo method (see Sec. 3). A discussion and comparison of the results in Sec. 4 completes the presentation.

2. BOSON HUBBARD MODEL IN THE MEAN-FIELD APPROXIMATION

A qualitative approximation for the phase diagram of the model (1) can be obtained using an approach described previously.^{19,23,24} In terms of this model, the boundary of the ordered state is located where the local density of the superconducting component vanishes in the effective functional describing long-wavelength excitations of the system. The latter can be obtained in the usual way using the Hubbard–Stratonovich transform²⁵ followed by an expansion of the effective functional in components of the fluctuating field.^{19,24} The condition that the local density of the superconducting component vanish for the system described by the Hamiltonian (1) yields an equation for determining the boundary of the ordered state:

$$\tilde{q}^2 = \frac{4 \sum_{n=0}^{\infty} \left[\exp \left\{ -0.5 \tilde{q}^2 (n - \tilde{\eta})^2 / \tilde{T} \right\} - \exp \left\{ -0.5 \tilde{q}^2 (n + 1 - \tilde{\eta})^2 / \tilde{T} \right\} \right] (n + 1) / (2(n - \tilde{\eta}) + 1)}{\sum_{n=0}^{\infty} n \exp \left\{ -0.5 \tilde{q}^2 (n - \tilde{\eta})^2 / \tilde{T} \right\}} \quad (2)$$

Here we have used the following independent dimensionless parameters, which specify the state of the system:

$$\tilde{q} = \sqrt{U/t}, \quad \tilde{T} = \frac{k_B T}{t}, \quad \tilde{\eta} = \frac{\mu}{U} - \frac{2t}{U}$$

The average number of particles in the granules of the system in the disordered state is given by the equation

$$n_0 = \frac{\sum_{n=0}^{\infty} n \exp \left\{ -0.5 \tilde{q}^2 (n - \tilde{\eta})^2 / \tilde{T} \right\}}{\sum_{n=0}^{\infty} \exp \left\{ -0.5 \tilde{q}^2 (n - \tilde{\eta})^2 / \tilde{T} \right\}} \quad (3)$$

The solid curves in Fig. 1a represent the phase diagram of the system (1) in the variables $\{\sqrt{U/t}, \mu/U\}$ obtained by solving Eq. (2) for different temperatures $k_B T/t$. The region of low values of U/t (low particle interaction energies) corresponds to the superconducting (S) state of the system.

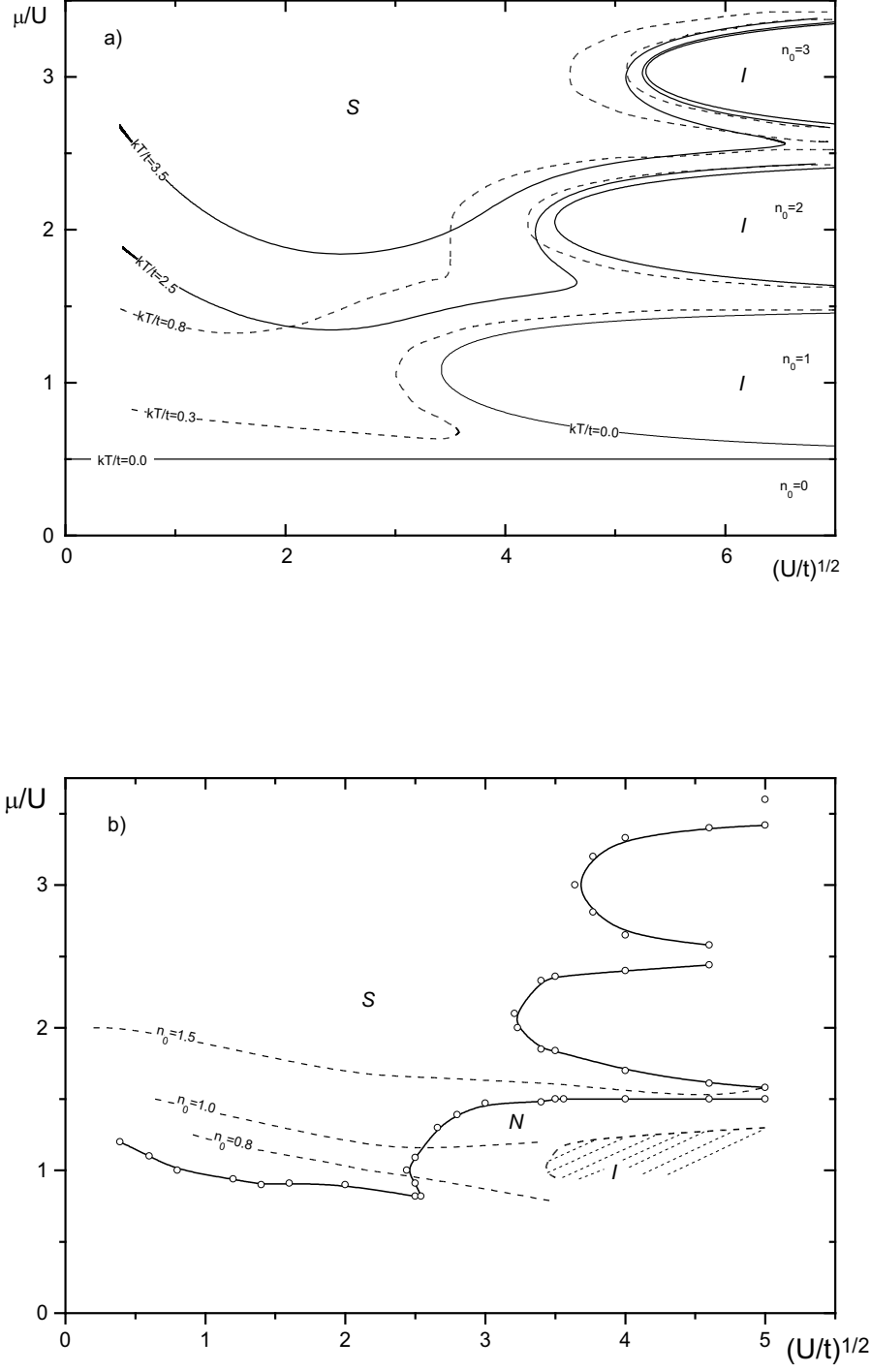


FIG. 1. Phase diagram for the Hubbard model (1) in the coordinates $\{\sqrt{U/t}, \mu/U\}$. S — the superconducting state; N — the normal (metallic) state; I — Mott insulator (hatched region in Fig. b). (a) Mean-field calculation. The solid curves were obtained by solving Eq. (2) and correspond to vanishing of the local density of the superfluid component. The Kosterlitz–Thouless topological transition (6) takes place on the dashed curves. (b) Monte Carlo calculations for $k_B T/t = 0.8$.

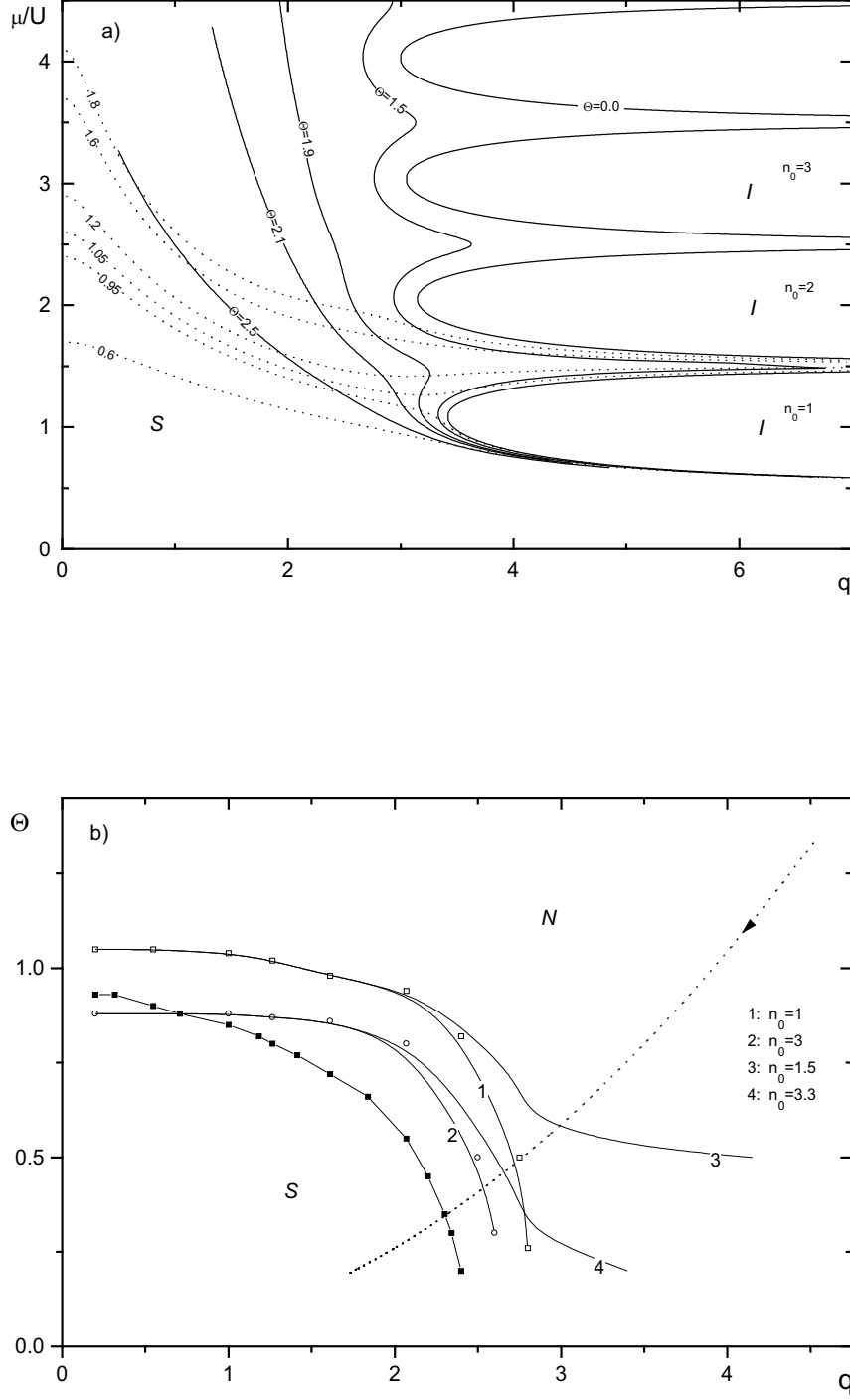


FIG. 2. Phase diagram for the Hubbard model (1) in the coordinates $\{q, \mu/U\}$ and $\{q, \Theta\}$. (a) Mean-field theory calculation. The solid curves were obtained by solving Eq. (2). The dotted curves correspond to the joint solution of the system of Eqs. (2) and (3). (b) Monte Carlo calculations. The filled squares are the phase diagram for the 2 + 1-dimensional XY model (for integer values of $n = k \gg 1$).³⁴ The system with $\{U/t, k_B T/t\} = \{3.5, 0.8\}$ and a variable chemical potential μ/U moves along the dotted curve (see Figs. 4 and 5 below).

As $k_B T/t \rightarrow 0$, the disordered state of the system corresponds to an integer average number of particles per

granule, $n_0 = k$, and determines the domain of existence of the Mott insulator (I).¹⁷ In this limit, the superfluid state of the system corresponds to the case of incommensurate populations, i.e., to a noninteger average number of particles per granule. Figure 1a shows that for $T = 0$ and half-integer values of the chemical potential $\mu = 0.5 + k$, a superconducting state exists for arbitrarily strong interparticle interactions, in accord with earlier works.^{17,18}

At finite temperatures an increase in the boson interaction force leads the system into a disordered state for arbitrary values of the chemical potential (see Fig. 1a). It is also clear from the figure that as the temperature is increased, the domain of existence of the ordered state is shifted toward higher values of the chemical potential.

The transition to the case of a system of macroscopic granules corresponds to increasing the particle density n_0 and reducing the role of the fluctuations in the modulus of the order parameter. It is most convenient to follow the changes in system (1) with increasing n_0 in the $\{q, \Theta\}$ plane, where we use the dimensionless temperature $\Theta = k_B T / t n_0$ and the quantum parameter $q = \sqrt{U / t n_0}$, which are the control parameters that also determine the state of the quantum XY model. The corresponding phase diagram is shown in Fig. 2a. As can be seen from this figure, for any values of U the estimate of the boundary of the ordered state in the Hubbard model according to mean-field theory lies above the corresponding limit in the XY model and approaches it as the average population n_0 of the lattice sites increases. Our calculations confirm that the phase diagram is periodic in the parameter μ/U when $n_0 \gg 1$.¹⁴

The dotted lines in Fig. 2a comprise a family of curves, at whose points a system with density n_0 becomes disordered. The points where they intersect the $\Theta = \text{const}$ lines determine the phase diagram of a system in the $\{q, \Theta\}$ plane corresponding to n_0 particles per granule. A similar analysis shows that for an incommensurate boson density at low temperatures (see below), an ordered (superconducting) state of an array exists for arbitrarily large values of q , i.e., for arbitrarily large quantum fluctuations in the phase of the superconducting order parameter in terms of the quantum XY model.

This approach only yields a qualitative estimate of the characteristic features of the phase diagram of this system. A comparison with the results of a numerical simulation (see below) shows that Eqs. (2) and (3) give a greatly overestimated value for the disorder temperature $\Theta_c(q; \mu/U)$. In order to obtain more accurate quantitative estimates, it is necessary to determine the temperature at which the global (rather than local, as in the method described above) superfluid density of the array vanishes. This temperature can be estimated as the Kosterlitz–Thouless temperature for the topological phase transition according to the classical XY model, onto which the initial system is mapped by expanding the effective Ginzburg–Landau functional for weak fluctuations in the phase of the order parameter.

Using the approach in Refs. 19, 24, and 26 it is easy to show that the effective action of the classical two-dimensional XY model sought is given by

$$S(\{\varphi_{\vec{k}}\}) = \frac{J_{XY}}{2} \sum_{\vec{k}} |\vec{k}|^2 \varphi_{\vec{k}} \varphi_{-\vec{k}} \approx J_{XY} \sum_{\langle i, j \rangle} (1 - \cos(\varphi_i - \varphi_j)), \quad J_{XY}(\mu/U; t/U; k_B T/t) = \frac{t \Delta^2}{4}, \quad (4)$$

where J_{XY} is the coupling constant for the effective XY model, which depends on the dimensionless parameters μ/U , t/U , and $k_B T/t$. The local superfluid density $\Delta^2/4$ of the system is described by the relation

$$\Delta = \frac{\text{tr} \{ (\hat{a}^\dagger + \hat{a}) e^{-\beta \hat{H}_{mf}} \}}{\text{tr} \{ e^{-\beta \hat{H}_{mf}} \}}, \quad \hat{H}_{mf} = \frac{U}{2} \hat{n}^2 + (2t - \mu) \hat{n} - t \Delta (\hat{a}^\dagger + \hat{a}) \quad (5)$$

A plot of the Kosterlitz–Thouless phase-transition temperatures in the effective XY model²⁷ specifies the boundary sought of the superfluid state for this array of granules:

$$k_B T^c = 0.98 J_{XY}(\mu/U; t/U; k_B T^c/t) \quad (6)$$

Estimates given by Eqs. (4) and (5) are shown as dashed curves in Fig. 1a. Note that, although the two approaches discussed above yield a similar qualitative behavior of the boundary of the ordered phase, the temperature of the topological phase transition in the effective XY model (6) is considerably lower than the temperature at which the local superfluid density vanishes according to Eq. (2). A comparison of the phase diagram obtained in this way with the results of Monte Carlo calculations (see below and Fig. 1) shows that they are in fair quantitative agreement.

3. QUANTUM MONTE CARLO METHOD. MEASURABLE QUANTITIES

The Trotter discretization procedure makes it possible to estimate all the thermodynamic averages of the observables of a D -dimensional quantum system in terms of a classical $D + 1$ -dimensional system, where the product of the matrix

elements (calculated approximately) of the high temperature density matrix serves as the Boltzmann weight of the configurations of the corresponding classical system. For studying the properties of the model (1), we shall use the “checkerboard version” of the quantum Monte Carlo method. (A detailed discussion of the discretization procedure and the organization of the Monte Carlo step during the simulation of systems of lattice bosons in a large canonical ensemble is given elsewhere.²⁸) In this method, the degrees of freedom of the discretized system are the occupation numbers $\{n_i^p\}$ of the sites of the $N \times N \times 4P$ three-dimensional lattice formed by $4P$ -fold multiplication of the initial $N \times N$ lattice along the imaginary time axis. The number of subdivisions P was chosen so that the parameter $\epsilon = q^2/P^2\Theta^2$, which characterizes the discretization error, would be less than 0.06.

The density ν_s of the superfluid component was calculated at each computational point of the phase diagram, whose position is specified by the parameters $\{\sqrt{U/t}, k_B T/t\}$ and the chemical potential μ/U . To find this quantity, we used both the fluctuations in the topological winding number^{20,28} and the correlation function of the paramagnetic current.²⁹ We found that when the average particle density at the boundary $n_0 < 2$ and $q > 2$, the statistical errors in the second method were considerably higher than the errors in determining the superfluid density from the fluctuations in the winding number, rendering it unsuitable.

We also measured n_0 , which is controlled by the chemical potential of the system, and the compressibility modulus κ , which is defined as

$$\kappa = k_B T \partial n_0 / \partial \mu = \frac{1}{4PN^2} \left\langle \sum_{p=0}^{4P-1} \sum_i (n_i^p)^2 \right\rangle - (n_0)^2 \quad (7)$$

It turned out to be convenient to make the measurements with $U/t = \text{const}$ and $k_B T/t = \text{const}$ and variation of the chemical potential μ/U . As an example, in Fig. 2b the line along which the system moves for $\{U/t, k_B T/t\} = \{3.5, 0.8\}$ is plotted in the coordinates $\{q, \Theta\}$. The location of the system on this line for a given value of the chemical potential can be determined by measuring the average number n_0 of particles per granule. In addition, we performed a number of calculations for fixed n_0 (in a canonical ensemble). It might be expected that for the same particle density n_0 the results of the simulation would be independent of the choice of the ensemble for sufficiently large systems. We tested this assumption and found for a system of dimension $N \times N = 6 \times 6$ that the difference in the measured quantities is less than 10% in the region of variation of the control parameters of interest to us. Therefore, in analyzing the results, data obtained by assigning the density n_0 (within a canonical distribution) and by assigning the chemical potential (which corresponds to using a grand canonical ensemble) can be used simultaneously.

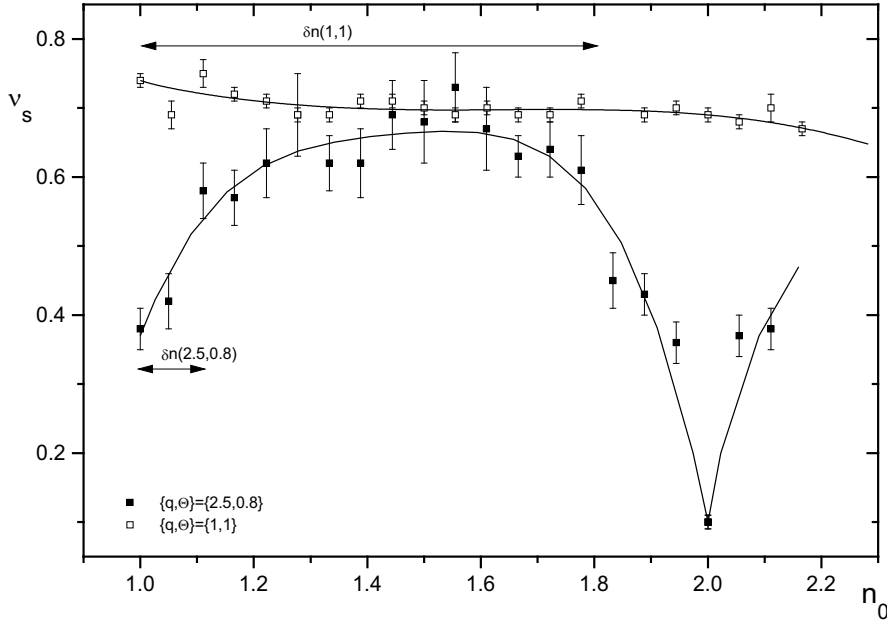


FIG. 3. Plots of the fraction ν_s of the superfluid component as a function of the average occupation number n_0 . The solid curves were obtained by interpolation of the data with a fourth-order polynomial. The lengths of the horizontal arrows correspond to estimates of $\delta n(q, \Theta) = \kappa(q, \Theta)|_{n_0=1}$. See Eq. (8).

4. DESCRIPTION AND DISCUSSION OF RESULTS

We first consider the computational results for $q < 1.5$, where the interparticle interaction is weak and mean-field theory (see Figs. 1a and 2a) predicts a monotonic dependence of the phase-transition temperature on the particle density n_0 . Figure 3 shows the calculated density of the superfluid component as a function of average occupation number, $\nu_s(n_0)$, for $q = 1$ and $\Theta = 1$ (unfilled symbols). As the occupation number increases, ν_s approaches a constant equal to the helicity modulus $\gamma(q, \Theta)$ in the $2 + 1$ -dimensional (quantum) XY model.^{30,31} We found previously²³ that in this region of the phase diagram, this limit, which corresponds to a small contribution of the quantum fluctuations to the modulus of the order parameter, is approached already when $n_0 = 4$ –5. The monotonicity of $\nu_s(n_0)$ suggests that the results of experiments on a system of isolated granules will not differ greatly from those on a system of granules on a substrate with an applied potential. It turned out that the system behaves similarly up to $q \simeq 2.3$. As the quantum parameter q is raised further and the temperature Θ is lowered, $\nu_s(n_0)$ ceases to be a monotonic function of the average occupation number n_0 . Characteristic oscillations in $\nu_s(n_0)$ with minima at integer values of $n_0 = k$ are noticeable in Fig. 3 ($\{q, \Theta\} = \{2.75, 0.5\}$, filled symbols). For sufficiently high boson densities ($n_0 > 7$ in the region $\{q, \Theta\} = \{2.5, 0.5\}$; see Ref. 23) there is a transition to the quasiclassical limit and the density of the superconducting component $\nu_s(n_0)$ becomes a periodic function of the average occupation number with a period of unity.¹⁴

Figure 4a shows plots of the superfluid density as a function of the chemical potential of the system, which is proportional to the voltage applied to the substrate. The squares refer to a system with $\sqrt{U/t} = 2$, $k_B T/t = 0.8$ and the triangles to $\sqrt{U/t} = 2.5$, $k_B T/t = 0.8$. The feature at $n_0 \simeq 1$ [a dip on the plots of $\nu_s(\mu/U)$ and $\kappa(\mu/U)$ for at $\mu/U \simeq 1.2$] is poorly seen for $q \simeq 2$ (squares in Fig. 4) but becomes clearly evident for $q \simeq 2.5$ (triangles). The figure confirms that the case of commensurate populations corresponds to lower densities of the superfluid component, i.e., increasing the deviation of the average boson density from integer values of $n_0 = k$ leads to spreading of the phase diagrams for the model (1) in the $\{q, \Theta\}$ plane.

As the quantum parameter q is increased further, the differences in the properties of the system for integer and noninteger boson densities become increasingly more significant. In fact, based on the results from mean-field calculations (see Fig. 1a), we can assume that there is a value of U/t for which the $\sqrt{U/t} = \text{const}$ line intersects the region of the disordered state with $n_0 \simeq 1$. Further increases in the interaction constant U should lead to the possibility of an intersection with the disordered region having $n_0 \simeq 2$, etc. The computational results shown in Fig. 5 confirm this assumption. Note that for $\sqrt{U/t} = 5.0$ (triangles in Fig. 5) and integer values of μ/U , changes in the chemical potential lead to essentially no change in the average number of particles at the sites in the system. This feature, which is characteristic of an insulator, can also be seen in Fig. 5b, which shows a plot of the compressibility modulus κ as a function of the chemical potential μ/U .

At $T = 0$ a lattice boson system without disorder undergoes a superconductor-insulator phase transition.¹⁷ As for the system being studied here, one can assume that as the temperature is raised, the domain of existence of the Mott insulator decreases and shifts toward larger U/t , so that, for example, on moving along the $\mu/U = k$ line the superfluid phase is replaced by the normal (metallic) phase. Further increases in the interaction constant, under which the magnitude of the Mott dielectric gap increases so much that thermal excitations become unimportant, lead to crossover formation of the insulator state.³² An estimate of the location of the boundary of the insulator state for $n_0 = 1$, based on our calculations, is shown in Fig. 1b. Several constant-density ($n_0 = \text{const}$) contours are also shown for comparison. A more detailed study of crossover-type metal-insulator transitions requires that systems of substantially larger size be examined.^{24,33,34}

An analysis of Figs. 1–5 shows that interesting effects caused by incommensurate populating of the sites in the system occur only in quite strongly interacting systems and at sufficiently low temperatures. For the boson Hubbard model (1), the domain in which they exist can be estimated by the inequalities $q > 2.3$ and $\Theta < 0.7$. Quantitatively, the magnitude of the deviation δn of the average particle density from an integer value of $n_0 = k$ at which a significant change in the system properties will be observed can be expressed in terms of the compressibility modulus:

$$\delta n \approx \left. \frac{\partial n_0}{\partial \mu} \right|_{n_0=k} k_B T = \kappa|_{n_0=k} \quad (8)$$

It is known that the compressibility modulus [which is inversely proportional to the phase fluctuations in the $2 + 1$ -dimensional XY model; see Eq. (7)] falls off substantially as the quantum parameter q becomes greater and the temperature Θ is reduced. We found that the following estimates hold for the Hubbard model (1) when $n_0 = 1$: $\delta n \approx 1.2$ for $\{q, \Theta\} \approx \{0.5, 1\}$, $\delta n \approx 0.3$ for $\{q, \Theta\} \approx \{2.5, 0.8\}$, and $\delta n \approx 0.06$ for $\{q, \Theta\} \approx \{2.75, 0.5\}$. As an illustration, Fig. 3 shows the values of $\delta n(q, \Theta)$ found for the points $\{q, \Theta\} = \{1, 1\}$ (unfilled symbols) and $\{q, \Theta\} = \{2.5, 0.8\}$ (filled symbols). The figure demonstrates the fair agreement between the theoretical estimate (8) and the Monte Carlo calculations.

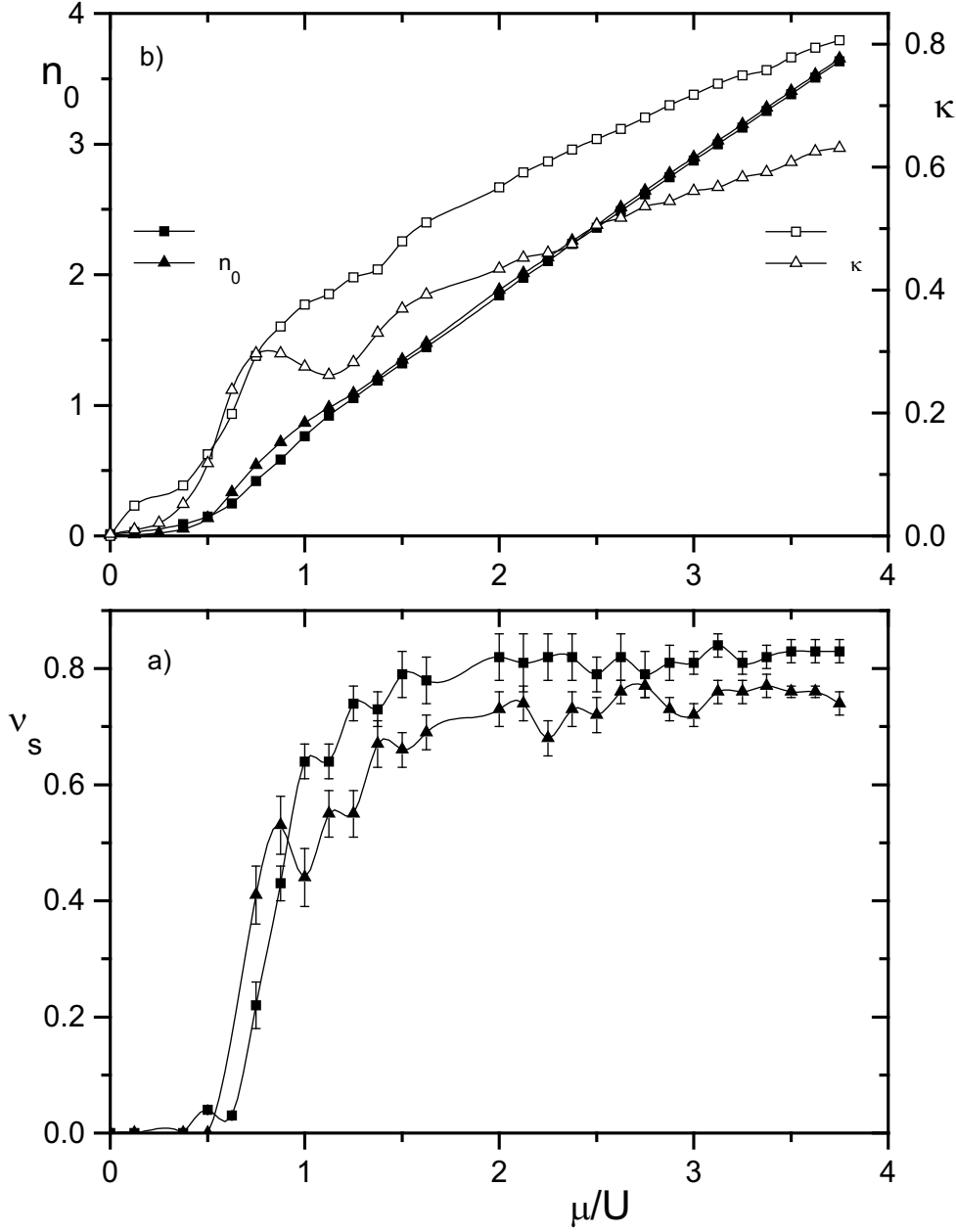


FIG. 4. (a) The superfluid density ν_s as a function of the chemical potential μ/U . (b) Average particle density n_0 (filled symbols) and compressibility modulus κ (unfilled symbols) as functions of the chemical potential μ/U : squares — $\{U/t, k_B T/t\} = \{2.0, 0.8\}$, triangles — $\{U/t, k_B T/t\} = \{2.5, 0.8\}$. Spline interpolations are shown for visual convenience. When not indicated the statistical errors are smaller than the sizes of the corresponding symbols.

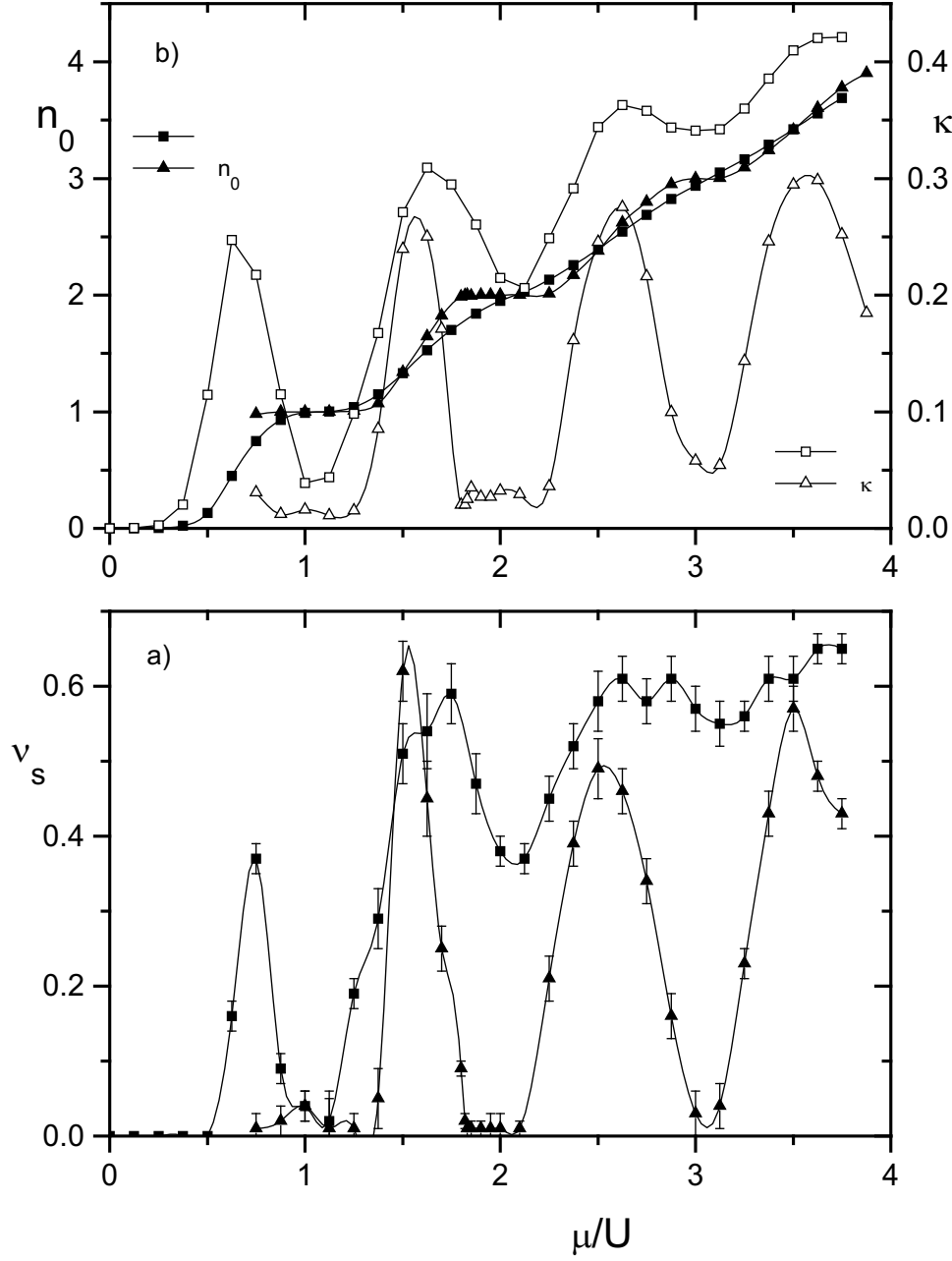


FIG. 5. (a) The superfluid density ν_s as a function of the chemical potential μ/U . (b) Average particle density n_0 (filled symbols) and compressibility modulus κ (unfilled symbols) as functions of the chemical potential μ/U : squares — $\{U/t, k_B T/t\} = \{3.5, 0.8\}$, triangles — $\{U/t, k_B T/t\} = \{5.0, 0.8\}$.

The above results yield the phase diagram of the system shown in Fig. 1b (in the coordinates $\{\sqrt{U/t}, \mu/T\}$ for $k_B T/t = 0.8$) and in Fig. 2b (in the coordinates $\{q, \Theta\}$). The location of the boundary of the ordered superconducting state was estimated from the universal jump in the superfluid density³⁰ and from the location of the peak in its temperature derivative.³⁵

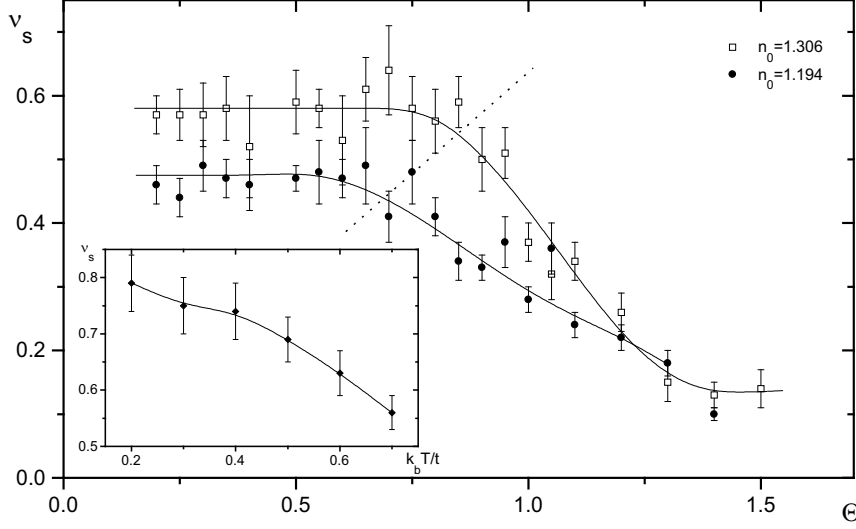


FIG. 6. Superfluid density of the system as a function of $\Theta = k_B T / t n_0$ for $q = 3.0$; unfilled symbols — $n_0 = 1.306$; filled symbols — $n_0 = 1.194$. The dotted line indicates the straight line $\nu_s = 2\Theta/\pi$. The inset shows the calculated superfluid density of the system as a function of $k_B T/t$ for $\sqrt{U}/t = 2.5$ and $\mu/U = 0.75$.

Quantum phase transitions (at $T = 0$) in the two-dimensional Hubbard model (1) are determined by the critical properties of the corresponding effective three-dimensional system.^{17,32} At finite temperatures a Josephson array will display Kosterlitz–Thouless critical behavior on some $\Theta_c(\mu/U, q)$ curve. We now try to evaluate the effect of quantum fluctuations on the temperature of this transition, assuming it to be rather low (see below). There is interest in two cases: (a) the system has a commensurate particle density $n_0 = k$ and at some temperature $\Theta_c(q)$ it undergoes a transition from the superconducting to the normal state, and (b) a phase transition takes place at $\Theta_c(n_0)$ because of a change in the density n_0 .

For the region near the point $q_c^{XY} \approx 2.5$ of the quantum transition in the $2 + 1$ -dimensional XY model (see Fig. 2b), the temperature $\Theta_c(q)$ of the topological Kosterlitz–Thouless phase transition has been estimated²⁴ as $\Theta_c(q) \sim |q - q_c^{XY}|^{\zeta_1}$ with $\zeta_1 \approx 0.67$. This estimate was derived under the assumption that the system temperature is less than the temperature of the $2D \rightarrow 3D$ crossover; i.e., $\Theta \leq \Theta_{3D}$, where $\Theta_{3D} \sim |q - q_c^{XY}|^\nu$ with $\nu \approx \zeta_1$. This makes it possible to estimate the density of the superfluid component, $\nu_s(q, \Theta)$, which determines the phase-transition temperature $\Theta_c(q)$, as $\nu_s(q, \Theta) \approx \nu_s(q, 0)$. Evidently, similar arguments apply in the neighborhood of the points $q_c^H|_{n_0=k}$ of the quantum phase transitions in the Hubbard model (1) for integer populating of the array granules, i.e., $n_0 = k$ ($q_c^H|_{n_0=1} \approx 2.8$; see Ref. 20). Thus, we may expect that

$$\Theta^c(q; k) \sim |q_c^H|_{n_0=k} - q|^{0.67} \quad (9)$$

Similar arguments can also be applied to the case of incommensurate boson densities, $n_0 \neq k$. It has been shown^{17,20} that $\nu_s \sim |n_0 - k|^{\zeta_2}$ with $\zeta_2 \approx 1.0$ for $q > q_c^H$. Thus, the following relation holds for the temperature $\Theta_c(n_0)$ of the topological Kosterlitz–Thouless phase transition:

$$\Theta^c(n_0) \sim |n_0 - k|^{1.0} \quad (10)$$

Our quantum calculations are in qualitative agreement with the predictions of Eqs. (9) and (10), but it is difficult to confirm their validity with sufficient accuracy because of the large errors in determining the position of the quantum phase-transition line $\mu(U/t; T = 0)$.

There is great interest in the question of the existence of reentrant superconductivity, for which, within some range of variation of the quantum parameter q , disorder sets in not only as the temperature Θ is raised, but also as it is lowered. The existence of reentrant effects has been predicted a number of times within the quantum XY model (see

Ref. 13 and the literature cited therein), but as far as we know, computer simulation cannot unequivocally confirm^{31,36} or refute³⁴ the existence of this phenomenon. The earlier numerical calculations of the Hubbard model did not reveal low-temperature instability or reentrant superconductivity.^{20,23} For this paper we studied the low temperature region $q \approx 2.5$, $\Theta < 0.5$ of the boson Hubbard model both for different noninteger occupation numbers and for fixed values of the chemical potential. The results, shown in Fig. 6, indicate a lack of reentrant superconductivity effects, at least within the range of variation of the control parameters studied here. Special attention was devoted to the region $\{\sqrt{U/t}, \mu/U\} \approx \{2.5, 0.8\}$, within which low-temperature disorder has been predicted.¹⁹ The numerical simulations clearly showed that there were no such effects within this region. In addition, our calculation of the boundary of the ordered state using mean-field theory (see Sec. 2) also disagrees with these predictions. Note that Eq. (2), which specifies the boundary of the ordered state, is more accurate in this method than is the equation used in the paper just cited inasmuch as its derivation did not rely on the assumption that $n_0 \gg 1$ [the latter is valid for $\mu/U \gg 1$ (Ref. 19)].

To conclude, we have analyzed the effect of quantum fluctuations in the phase and modulus of the superconducting or superfluid order parameter on the character of the ordering in two-dimensional mesoscopic Josephson and granular systems within a lattice boson Hubbard model. Quantum Monte Carlo calculations have been used to show that the way the system properties change as a result of modulation of the average occupation number of the array elements by the chemical potential (the substrate potential) is determined by the parameter $q = \sqrt{U/t}n_0$ (i.e., the ratio of the characteristic Coulomb energy of a granule to the Josephson tunneling energy). For $q < 1.5$, which is the quasiclassical region for the quantum XY model and the region of strong fluctuations in the modulus of the order parameter for the Hubbard model (1), the system properties are insensitive to the average number of particles in the granules. In the region where there are significant quantum fluctuations in the order parameter ($q > 2$, $\Theta < 0.8$), we have found that the state of the system depends (more distinctly at lower temperatures) on the average number of particles in it.

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¹ Two-dimensional Josephson arrays (of mesoscopic elements) with superfluid helium can, in principle, be formed by creating the corresponding “picture” in cesium on the substrate (since cesium is not wetted by helium).¹

² The interference of two Bose condensates has recently been studied.¹¹

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